Random Graphs
Exercise Sheet 6

Question 1. Suppose we flip n biased coins, each which land heads with probability p and tails with probability $1-p$, independently of the others. Let X be the number of heads flipped, let Z_i be the result of the ith coin flip and consider the martingale given by $X_i = \mathbb{E}(X|\sigma(Z_1,\ldots,Z_i))$. Using the Azuma-Hoeffding inequality bound, for any $t \geq 0$, the probability

$$\mathbb{P}(|X-\mathbb{E}(X)| \geq t).$$

Compare this to the Chernoff bounds.

Question 2. Let X be the number of triangles in $G_{n,p}$. For what values of p does it hold with exponentially high probability that

$$X = (1+o(1))\mathbb{E}(X)?$$

Question 3. Let p be fixed, $\varepsilon > 0$, $b = \frac{1}{1-p}$ and let $k = (2-\varepsilon)\log_b n$. Let Y be the largest size of a collection of vertex disjoint independent sets of size k in $G_{n,p}$ and let $\mathcal{K}$ be the collection of all independent sets of size k in $G_{n,p}$. By choosing a random subset of $\mathcal{K}$ and using the alteration method, show that

$$\mathbb{E}(Y) \geq (1+o(1))\frac{n^2}{4k^4}.$$

Deduce that

$$\mathbb{P}(\alpha(G_{n,p}) < k) \leq e^{-\tilde{\Omega}(n^2)},$$

where $\tilde{\Omega}$ means up to polylog factors.

Question 4. Let $p \geq \frac{(1.001)}{n}$. Show that

$$\mathbb{P}(\chi(G_{n,p}) \geq 3) \geq 1 - \frac{1}{\log n}.$$

Question 5. Let $\mathrm{tcl}(G)$ be the largest t such that G contains a subdivision of a complete graph K_t. Show that with high probability

$$\mathrm{tcl}(G_{n,1/2}) = O\left(\sqrt{n}\right).$$

(* Show that in fact with high probability $\mathrm{tcl}(G_{n,1/2}) = \Theta\left(\sqrt{n}\right)$)

Question 6. Let $\mathrm{ccl}(G)$ be the largest t such that G contains a K_t minor. Show that with high probability

$$\mathrm{ccl}(G_{n,1/2}) = \Theta\left(\frac{n}{\sqrt{\log n}}\right).$$